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Borel convergence of the variationally improved mass expansion and dynamical symmetry breaking

J.-L. Kneur and D. Reynaud

Physique Mathématique et Théorique, UMR-5825-CNRS,
Université Montpellier II, F–34095 Montpellier Cedex 5, France.

Abstract

A modification of perturbation theory, known as delta-expansion (variationally improved perturbation), gave rigorously convergent series in some $D = 1$ models (oscillator energy levels) with factorially divergent ordinary perturbative expansions. In a generalization of variationally improved perturbation appropriate to renormalizable, asymptotically free theories, we show that the large expansion orders of certain physical quantities are similarly drastically improved, and prove the (Borel) convergence of the corresponding series. We argue in particular that non-ambiguous estimates of quantities relevant to dynamical (chiral) symmetry breaking in QCD, are possible in this resummation framework.

1 Introduction

A "first principle" determination of the order parameters characterizing dynamical (e.g. chiral) symmetry breaking (χSB) in asymptotically free theories (AFT) like QCD is traditionally considered inaccessible (except perhaps from lattice calculations), due to three main obstacles:
(i) order parameters, like the quark condensate $\langle \bar{q}q \rangle^{1/3}$, are expected to be of $\mathcal{O}(\Lambda_{QCD})$, so that the coupling at such scale is large: ordinary perturbative expansion is invalidated.
(ii) At arbitrary perturbative order, $\langle \bar{q}q \rangle$ and other χSB order parameters vanish anyhow in the massless limit: their chiral limits are (perturbatively) trivial.
(iii) A more subtle but equally important argument is that, attempts to extract genuine non-perturbative contributions to such quantities meet inherent ambiguities, as indicated by the (infrared) renormalon singularities of perturbative expansions[1, 2]. Conventional wisdom thus treats $\langle \bar{q}q \rangle$ and other non-perturbative condensates as *parameters* of a systematic operator product expansion (OPE)[1], as best illustrated in the SVZ formalism[3].
Yet in many field theory models, definite non-perturbative results may be obtained from an appropriately resummed (but different) expansion, like the $1/N$ expansion[4, 5]. There also exist powerful summation techniques, like the Borel method[6, 2] which, even for non Borel-summable expansions like in QCD typically, gives nevertheless precious informations on the nature of (power-like) non-perturbative contributions to a given physical quantity. An alternative summation method, known as delta-expansion (DE) or "variationally improved perturbation" (VIP)[7, 8], is based on a reorganization of the interaction Lagrangian to depend on arbitrary adjustable parameters, to be fixed by some optimization prescription. In various models DE-VIP exhibits (though often rather empirically) an improved convergence of perturbative expansion. Moreover in some $D = 1$ models, the anharmonic oscillator typically, DE-VIP is equivalent[9] to the "order-dependent mapping" (ODM) method[10], and optimization is equivalent to a rescaling of the adjustable oscillator frequency (mass) with perturbative order, which can essentially suppress the factorial asymptotic behaviour of ordinary perturbative coefficients. Such a procedure was proven rigorously to converge[9, 11] (for an adequately rescaled mass) toward the exact result, e.g. for the oscillator energy levels[12] and related quantities.

Here we reconsider a variant of DE-VIP adapted to higher dimensional renormalizable theories, proposed some time ago[13]–[15]. The basic idea is to perform a modification of perturbative expansions in two stages: first exploiting specific renormalization group (RG) properties, which transform the ordinary expansion (in a coupling g) of certain physical quantities, depending only on g and on a mass m, in the alternative form of "mass power" expansions (MPE) in $(\hat{m}/\Lambda)^\alpha$ [$\hat{m}$ is the renormalization scale-invariant mass, Λ the basic RG scale and α is given by known RG coefficients]. This construction resums RG dependence to all orders (at least in specific schemes), and most interestingly exhibits a non-trivial massless (chiral) limit[14, 15] for DSB (χSB) order parameters, or for analogous quantities like the "mass gap" in $D = 2$ models[13], thus circumventing obstacle (ii) above. Though it may look satisfactory to obtain a dynamical mass from pure RG considerations, such a result turns out to be well-defined only in the approximation of neglecting all the purely perturbative (non RG) dependence. When arbitrary large orders of the complete (non-log) perturbative series are included, our naive

[1]Unlike the gluon condensate, the presence of χSB condensates like $\langle \bar{q}q \rangle$ in OPE's is however not directly inferred by infrared renormalons, these being screened by chiral symmetry[1], cf. argument (ii) above. We will see that renormalons and argument (iii) are nevertheless relevant to χSB quantities in our context.

mass gap result is plagued with ambiguities originating mainly from renormalon singularities, cf. point (iii) above, as we shall examine in more details here.

However, in a second stage, an appropriate version of the (order–dependently rescaled) DE-VIP can be performed on the complete MPE series in $\hat{m}/\Lambda$, essentially replacing the true physical mass by an arbitrary adjustable mass parameter. In this note we mainly investigate the large order behaviour of the resulting "variational" expansion in this mass parameter[2]. We find that it produces a renormalization scheme (RS) dependent factorial *damping* of the original perturbative coefficients at large orders. Yet, unlike the oscillator case, the damping is generally not sufficient to make the DE-VIP series readily convergent, when the generically expected renormalon singularities are taken into account. But we show that it is enough to make it *Borel* convergent, at least in a certain class of RS. These results apply formally a priori to any (asymptotically free) renormalizable models. Some typical examples are the D=2 $O(N)$ Gross-Neveu (GN) model[5] (where the mass gap is known exactly[17]); or a $D = 4$ gauged AFT with n_f massless fermions like QCD, where the expected[18] $SU(n_f)_L \times SU(n_f)_R \to SU(n_f)_V$ breaking is exhibited via non-perturbative order parameters.

2 Transmuted mass expansion and mass gap

In this and next section we summarize some of the construction in [13]–[15], with a somewhat different (and perhaps, more transparent) presentation. To illustrate simply the first stage, consider in a "generic" AFT the first RG order evolution for the renormalized "current" mass:

$$M_1 = m(\mu)\,[1 + 2b_0 g^2(\mu)\ln(M_1/\mu)]^{-\frac{\gamma_0}{2\,b_0}}\ , \tag{1}$$

where b_0, γ_0 are one-loop RG coefficients, with $b_0 > 0$ for an AFT [our normalization is $\beta(g) = -b_0 g^3 - b_1 g^5 - \cdots$, $\gamma_m(g) = \gamma_0 g^2 + \gamma_1 g^4 + \cdots$], and the *self-consistent* condition $M_1 \equiv m(M_1)$ defines M_1. Now, equivalently Eq. (1) reads

$$M_1 \ = \hat{m}\,[\ln(M_1/\Lambda)]^{-A} \equiv \ \hat{m}\,F^{-A} \tag{2}$$

with $\Lambda = \mu\exp[-1/2b_0 g^2(\mu)]$ the RG invariant scale, $\hat{m} \equiv m(\mu)[2b_0 g^2(\mu)]^{-A}$ the scale invariant mass ($A \equiv \gamma_0/(2b_0)$), and in Eq. (2)

$$F(\hat{m}/\Lambda) \equiv \ln(\hat{m}/\Lambda) - A\ln F \ = A\,W[A^{-1}(\hat{m}/\Lambda)^{1/A}] \tag{3}$$

where the Lambert[19] function $W[x] \equiv \ln x - \ln W$, is plotted in Fig 1. Eq. (3) has the remarkable property: $F \simeq (\hat{m}/\Lambda)^{1/A}$ for $\hat{m} \to 0$, in contrast with the ordinary Log (see Fig. 1), however asymptotic to $F(\hat{m}/\Lambda)$ for $\hat{m} \gg \Lambda$. More precisely, on its principal branch (which is real-valued for real arguments), F has an alternative series expansion:

$$F(x) = \sum_{p=0}^{\infty}(\frac{-1}{A})^p \frac{(p+1)^p}{(p+1)!}\,x^{\frac{p+1}{A}} \tag{4}$$

of finite convergence radius $R_c = e^{-A}A^A$. $M_1(\hat{m})$ in Eq. (2) thus exhibits different branches according to the values of the RG parameter A (see Fig. 2). Now, for most values of A, there

[2]See also ref. [16] for a preliminary discussion.

is only one branch which for *real* $\hat{m}$ values, is real and continuously matching the asymptotic perturbative behaviour of F at large $\hat{m}$: the one giving a non-zero "mass gap" $M_1 = \Lambda$ for $\hat{m} \to 0$ (region I in Fig. 2). Algebraically, the mass is obtained by expanding Eq. (4) in (2):

$$M_1(\hat{m} \to 0) = \hat{m}\,[(\hat{m}/\Lambda)^{1/A} + \cdots]^{-A} \;=\; \Lambda\,(1 + \mathcal{O}(\hat{m}/\Lambda)^{1/A}) \,, \tag{5}$$

which may be viewed as a generalization (for $m \neq 0$) of "dimensional transmutation". Note that Eq. (5) readily reproduces, e.g., the GN $O(N)$ model mass gap in the large N, $m \to 0$ limit (where $A \to 1$ for $N \to \infty$), traditionally obtained in a different way [5]. More generally, $F(\hat{m})$ provides an explicit bridge between the "non-perturbative" $\hat{m} \lesssim \Lambda$ regime, where F has power expansion (4), and the short distance perturbative $\hat{m} \gg \Lambda$ (Log) regime. A crucial point is the difference between the usual effective coupling $g^2(p^2) \equiv 1/[b_0 \ln(p^2/\Lambda^2)]$, having a Landau pole at $p^2 = \Lambda^2$, and $F^{-1}(\hat{m})$ here, having its pole at $\hat{m} = 0$, governing the massless limit (5) of the (pure RG) mass gap Eq. (2)[3]. Accordingly along the continuous branch I, $M_1(\hat{m})$ has no singularity for $0 < \hat{m} < \infty$, as is clear from Eq. (5) and Fig. 1,2.

3 Pole mass gap and other DSB quantities

Eq. (2) also defines a (lowest order) "pole" mass, being scale invariant to all orders (and gauge invariant as well, if gauge symmetry is relevant, as in QCD), thanks to its continued fraction form in M_1. Yet the genuine pole mass is not given simply by Eq. (1), as it includes non-log perturbative and RG contributions of arbitrary higher orders. There is a formal expansion relating the current mass $m(\mu \equiv M^{pole})$ and M^{pole} at arbitrary perturbative orders[21]:

$$M^{pole} = m(M^{pole})[1 + \sum_{n=1}^{\infty} c_n g^{2n}(M^{pole})\,] \tag{6}$$

where for most theories the series in brackets is unfortunately only known at present up to the second or third order coefficients c_2, c_3, like e.g. in the GN model[22, 13] or QCD[21]. Nevertheless a generalization of (2), perturbatively equivalent[15] to (6), can be defined[4]:

$$M^P(\hat{m}) = 2^{-C}\,\hat{m} F^{-A}[C+F]^{-B}\,\sum_{n=0}^{\infty} d_n\,(2b_0 F)^{-n}\,, \tag{7}$$

with

$$F = \ln[\hat{m}/\Lambda] - A \ln F - (B-C)\ln[C+F], \tag{8}$$

$$A = \frac{\gamma_1}{2b_1},\; B = \frac{\gamma_0}{2b_0} - A,\; C = \frac{b_1}{2b_0^2}\,. \tag{9}$$

$F(\hat{m})$ in (7), (8) resums the RG dependence in $\ln[\hat{m}]$ at two-loop order exactly (or even to all orders in the scheme $b_i = 0, \gamma_i = 0\ \forall i \geq 2$). Most interestingly, similarly to Eq. (4) F also has an (A, B, C dependent) expansion in $(\hat{m}/\Lambda)^{1/A}$ for sufficiently small $\hat{m}$, with A now defined in Eq. (9). The coefficients d_n implicitly include the non-log perturbative contributions

[3] $W(x)$ appears in various branches of physics[19], in particular recently also in the QCD and RG context[20]. Yet its connection with non-trivial chiral limit (5) was unnoticed before [13]–[15], to the best of our knowledge.

[4] Strictly, Eq. (7) applies only if $C \equiv b_1/(2b_0^2) \geq 0$. If $C < 0$ (as in the $O(N)$ GN model, corresponding to an infrared fixed-point at $g^2 = -b_0/b_1 > 0$), an alternative appropriate RG summation can be defined[13, 25].

from the n-loop graphs c_n in (6) (generically dominant, as discussed below), plus eventually (subdominant) contributions from higher RG orders.

A similar construction can be performed for other physical quantities, at least those depending only on m and g, i.e. having a perturbative expansion of the form

$$m^k \sum_{n,r} a_{n,r} \ln^{n-r}(m/\mu)\, g^{2n} \tag{10}$$

with k the appropriate mass dimension, which accordingly are vanishing in the massless limit. Examples are the perturbative expansion of the GN model vacuum energy[13], or in QCD the χSB order parameters F_π/Λ (the pion decay constant) and $\langle \bar{q}q \rangle(\mu)/\Lambda^3$[14, 15].

Now in (7), there are crucial differences with the "pure RG" mass gap, Eq. (2):

–The pole mass (or other physical quantities similarly) is infrared finite, gauge [23]–, scale– and scheme–invariant, but the relation between the pole mass and e.g. the running mass in (6) is scheme dependent, which is manifested here by the RS-dependence in (7) of the perturbative coefficients d_n, the RG coefficients A, B in Eq. (9), and of course Λ (the precise variation of these parameters under a general RS change is given e.g in [15]).

–Moreover, the dominant contributions d_n in (7) behave rather generically as[24, 2]:

$$d_{n+1} \underset{n \to \infty}{\sim} (2b_0)^n\, n! \tag{11}$$

so that the series Eq. (7) is badly divergent for any $\hat{m}$, and not even Borel summable: such a factorial growth of the perturbative coefficients, with no sign alternation, implies[2] ambiguities of $\mathcal{O}(\Lambda)$, as we reexamine within the present context in section 5. The $O(N)$ GN model mass gap, at order $1/N$, also exhibits infrared renormalons similar[25] to (11), if considering only its naive perturbative expansion. In QCD, insertions of the (resummed) gluon propagator in the F_π or $\langle \bar{q}q \rangle$ perturbative expressions potentially give factorially growing asymptotic coefficients: while usually considered irrelevant in the $m \to 0$ limit (cf. argument (ii) above), the factorial behaviour survives a priori in our construction due to the non-trivial chiral limit[5].

4 Variationally improved mass expansion

We shall examine now how to possibly cure the latter potential ambiguities of such a resummation of DSB quantities, by combining the previous MPE series construction leading to e.g. Eq (7) with a specific form of delta-expansion. As mentioned in introduction, DE-VIP is essentially a reorganization of the interaction terms of the Lagrangian. More specifically here, we define a (linear) DE as the substitution

$$m(\mu) \to (1-\delta)\, m_v;\; g^2(\mu) \to \delta\, g^2(\mu) \tag{12}$$

within perturbative expressions at arbitrary order, where $m(\mu)$ is the renormalized Lagrangian mass (in e.g. $\overline{MS}$ scheme), δ the new expansion parameter, and m_v an *arbitrary* adjustable mass. (12) is equivalent to adding and subtracting to the massless Lagrangian a "trial" mass term m_v [δ interpolating between the free ($\delta = 0$) and the interacting *massless* Lagrangian ($\delta = 1$)], and is entirely compatible with renormalization[13] and gauge-invariance[15]. The

[5]The precise form of those "χSB parameter renormalons" will be discussed elsewhere[25, 26].

procedure then usually[8] is to take the limit $\delta \to 1$ *after* performing a perturbative expansion of the relevant physical quantities to fixed order δ^k, exhibiting a residual m_v dependence, so that an optimization prescription, typically the "principle of minimal sensitivity" (PMS)[7], can be applied with respect to m_v. However, we go here a step beyond this standard PMS usage (whose eventual success in $D > 1$ models is often mostly empirical), by following more closely the logic that leads to rigorous convergence properties of the DE method for the oscillator.

In what follows we only investigate for simplicity the mass gap Eq (7), but our construction can easily be generalized to similar DSB quantities, of perturbative form (10). After applying substitution (12), $M^P(\hat{m}, \delta) \equiv \sum_k a_k(\hat{m})\delta^k$ can be most conveniently directly resummed, for $\delta \to 1$, by contour integration around $\delta = 0$, to arbitrary order K: an appropriate change of variable[13] allowing to study the $m(\mu) \to 0$ (equivalently $\delta \to 1$) limit in Eq. (12) is:

$$\delta \equiv 1 - v/K \; ; \quad m_v = K^\gamma \; \hat{m}_v \; . \tag{13}$$

Eq. (13) is simply a convenient way of parameterizing how rapidly the Lagrangian mass $m(\mu) \to 0$ limit is reached (as controlled by $\gamma \leq 1$) as function of the (maximal) delta-expansion order K. Similarly to refs. [9] the point is to adjust the rates at which $m(\mu) \to 0$ ($\delta \to 1$) and $K \to \infty$ are simultaneously reached, with no a priori need of invoking explicit optimization principle. Though the MPE series (7) is more involved than e.g. the oscillator energy level expansions [the reminiscence of RG Logs making the mass dependence, given by (3),(4), more involved than a single power of m], the freedom in rescaling with δ-expansion order the trial parameter m_v is similar, since the series (7), via F, depends only on $\hat{m}_v/\Lambda$.
The final contour integral summation takes a simple form, for $K \to \infty$:

$$M^P/\Lambda \sim \sum_{n=0}^{N} \frac{1}{2\pi i} \oint dv \; e^{(v/m")} \; F^{-A}[v] \, d_n \, (2b_0 F)^{-n} \tag{14}$$

($m" \equiv \hat{m}_v/\Lambda$), where after deformation the contour encircles the semi-axis $Re[v] < 0$ (see Fig. 3) and for simplicity we fix from now the scaling parameter in Eq. (13) to its maximal value ($\gamma = 1$) still compatible with massless limit (for $\hat{m}_v \to 0$). [The general γ scaling (13) can be analyzed[25, 26] in a way more similar to the oscillator [9, 11], i.e. without the peculiar contour δ-summation Eq. (14), but largely complicates the algebraic analysis for renormalizable theories. In (14) we also omit some overall constant factors (due e.g. to Λ definition) irrelevant for convergence properties, and temporarily made a RS choice such that $B \equiv 0$ in (7)–(9), rendering certain algebraic expressions below more tractable, without much loss of generality.]
Eq. (14) can be well approximated analytically (at least for slightly restricted RS choices, as indicated above and further below):

$$M^P/\Lambda \sim 1 + \frac{1}{2b_0} \sum_{q=1}^{N} \left[\sum_{p=0}^{N-q} \frac{\Gamma[p+q](p+q+A)(q+A)^{p-1}}{A^p \, \Gamma[1+p] \, \Gamma[1+q/A]} \right] (m")^{-q/A} \tag{15}$$

where we assumed the leading renormalon behaviour Eq. (11)[6], N is maximal perturbative order, and we used essentially Eq. (4) together with $\oint dv e^v v^r = 2\pi i/\Gamma[-r] \; \forall \, r$.
In fact, some restrictions apply to (15): first, the sum over p is bounded as given, iff

$$1/A \in \mathbf{N}^* \tag{16}$$

[6]The original $n!$ coefficients in Eq. (11) correspond to $\Gamma[p+q]$ in (15). Higher order refinements on infrared renormalon structure may easily be implemented: it essentially replaces $(n-1)! \to \Gamma[n + b_1/(2b_0^2)](1 + r_1/n + ..)$ where r_1 depends on b_1 etc [2], without affecting the convergence properties discussed below.

which we assume for simplicity from now. This is not much restrictive, except that for arbitrary AFT it is generally not possible both that A satisfies (16) *and* $B = 0$ in Eq. (9), as assumed in (14). But the more general scheme $B \neq 0$ simply makes Eq. (15) algebraically more involved, without affecting the asymptotic behaviour and convergence properties discussed below.
Second, strictly (15) is valid only asymptotically, for sufficiently large N: due to the finite convergence radius of expansion (4), interchanging the sum in (4) and integration in (14) is not rigorously justified. However, when (16) holds, the formerly branch point $v = 0$ is simply a pole, which allows to choose an equivalent contour of arbitrarily small radius around $v = 0$, thus always inside the convergence radius of (4) (see the dashed small circle contour in Fig. 3). So, only the simple pole terms v^{-1} contribute to Eq. (14), which finally sum up to (15). The extra contribution (around the cut at $v = -e^{-1}$, e.g. for $A = 1$) gives the difference between the "exact" integral (14) and expansion (15), and can be evaluated numerically. These contributions are easily shown for $A = 1$ to contribute as $\mathcal{O}(e^{-(e\, m'')^{-1}})h[N]$ relative to (15), where $h[N]$ rapidly decreases for $N \to \infty$. [If $A \neq 1$ and B arbitrary, contributions from extra cuts are not so simply estimated, and we were only able to check numerically that they are negligible with respect to (15) for sufficiently large N.] Thus for large enough N (and/or small m'') those contributions are unessential for the convergence properties discussed below.

The announced factorial damping of coefficients, as compared to the original perturbative expansion, is explicit in Eq. (15). Yet, closer examination indicates that the damping is insufficient to make this series for $N \to \infty$ readily convergent. Before considering the asymptotic behaviour of the full series (15), it is instructive to examine the $p = 0$ terms, behaving as:

$$\sim \sum_{q}^{N} \Gamma[q]/\Gamma[1 + q/A]\; (m'')^{-q/A}\;. \tag{17}$$

Let us indeed mention that a contour δ–summation similar to (13), (14), when applied to the oscillator energy levels, leads to a simple expansion precisely of the form (17) (upon the replacements $A \to 2/3$; $m'' \to m^2 g^{-2/3}$). It gives a strictly convergent series, for appropriate values of the scaling parameter: $1/3 \leq \gamma < 1/2$, in consistency with the results in [9].
Similarly, the denominators in (17) overcompensate the factorials in the numerators iff

$$0 < A \leq 1\;, \tag{18}$$

where for $D > 1$ AFT, A is RS dependent, as discussed in section 3. Thus, if our series would only consist of terms of the form Eq. (17), the solution would be simply to perform appropriate scheme changes $A \to A'$ in (7), (15) etc, so that a damping of coefficients larger than (or equal to) the factorial growth would make the series convergent. [For such RS changes in A one should consistently derive the corresponding change in e.g. the first few perturbative coefficients d_1, etc, and in Λ, but this one-parameter RS change does not reintroduce any factorial behaviour in the d_n at large order. Moreover, if (18) holds, any generic infrared (or ultraviolet) renormalon behaviour, of the form[2] $\sim r^n\, n!$ with r arbitrary, is damped similarly.]
Unfortunately, the large N behaviour of (15) differs from the simple "oscillator form" (17), due to the $p \geq 1$ terms in expansion (4) reminiscent of RG properties. For any low $p \ll N$, renormalon factorials are still overcompensated if $A \leq 1$, but the $\Gamma[1 + q/A]$ damping decreases in strength as p increases, giving increasing contributions to the sum over p. Now, another damping appears for $p \sim N - q$ terms: $\sim \Gamma[N]/\Gamma[1 + N - q]\; (m'')^{-q/A}$, but is clearly insufficient

to overcompensate the factorial in the numerator, and would become ineffective for $q \sim N$ (though in this case, the sum over p is bounded to small values, $p \ll N$). All in all, the leading contributions to the coefficients of (15) happen at intermediate values of p, and can be shown[25] to be of the form $(N!)^{s[A]}$ where $0 < s[A] < 1$. Nevertheless, the idea of damping factorials from appropriate RS choice does survive, and is sufficient to make the complete series Eq. (15) *Borel* summable, as examined in next section.

5 Borel convergence of DE-VIP

A Borel integral slightly adapted to our case reads[7]:

$$BI(\hat{m}) \equiv \tilde{M}^P(\hat{m}) = \hat{m}\; F^{1-A} \int_0^\infty dt e^{-F\,t}\; [1 + (2b_0\, F)^{-1} \sum_{n=0}^{\infty} t^n\,] \tag{19}$$

which would be equal to (7) by formal expansion (upon assuming Eq. (11)), would the pole at $t_0 = 1$ not make the integral (19) ill-defined. One should make a choice in deforming the contour e.g. above (or below) the pole, which results in an ambiguity, easily calculated to be $\mathcal{O}(e^{-F})$. Since $F \sim \ln[\hat{m}/\Lambda]$ for $\hat{m} \gg \Lambda$, an $\mathcal{O}(\Lambda/m)$ ambiguity[2] for the "short distance" $(M, \hat{m} \gg \Lambda)$ pole mass is recovered. But in our construction Eq. (4) allows to trace the behaviour of F all the way down to $\hat{m} \to 0$, where $F \to 0$: there the ambiguity becomes $\mathcal{O}(1)$, and the naive RG–summed mass gap (5), which is $\mathcal{O}(\Lambda)$, gets an ambiguity of same order, as announced.

Now for any given choice of contour avoiding the pole (or cut[2] at higher RG order) in the Borel plane t, let us apply the DE-VIP as defined in section 4, introducing the δ–expansion and contour resummation as in (14), this time on the Borel integral Eq (19). It gives:

$$\tilde{M}^P_{var}(\hat{m}_v) \sim \frac{1}{2i\pi} \oint dv e^v\; \frac{\hat{m}_v}{F^A} \int_0^\infty dt e^{-t}\; [1 + \frac{1}{2b_0} [(1 - t/F[v\,\hat{m}_v])^{-1}\; - 1]\,] \tag{20}$$

where the integrand is to be be understood as its expansion in t/F. Interchanging the two integrals in (20) and proceeding with algebraic manipulations similar to those leading to Eq.(15),

$$\tilde{M}^P_{var}(m") \sim \int_0^\infty dt e^{-t}\; [1 + \frac{e}{2b_0} (\frac{A+1}{A}) \sum_{q=1}^{\infty} \frac{(e\,/m")^{q/A}}{\Gamma[1+q/A]}\; t^q\,] \tag{21}$$

where the $N \to \infty$ limit of the Borel series coefficients $d^B_{q,N}(A)$ of $(m")^{-q/A}$,

$$d^B_{q,N}(A) \underset{N \to \infty}{\longrightarrow} \frac{A+1}{A}\; \frac{e^{1+q/A}}{\Gamma[1+q/A]}\;, \tag{22}$$

have been taken for simplicity. [$d^B_{q,N}(A)$ are similar to the coefficients in (15), but for the Borel-transformed series, i.e. with a factor $\Gamma[p+q]$ less. They can be simply expressed for arbitrary q, N in terms of incomplete Gamma functions]. On Fig. 4 we show the approximation (22) versus exact coefficients as function of the order q. The approximation is very good as long as $q \ll N$, and always bounds from above the true Borel integrand for any q, N.

[7] We define the Borel transform (integrand of (19)) by dividing series coefficients by $(n-1)!$ for convenience. Also, the summed RG-dependence $\hat{m}F^{-A}$, having no factorial behaviour, is factored out of the Borel transform.

For $A > 0$, it is then clear that the behaviour of the Borel integrand in (21) is that of an entire series, i.e. with *infinite* convergence radius, since there are no poles for $0 < t < \infty$. More precisely, the pole at $t_0 = 1$ in the original (standard) Borel integrand has been pushed to $t_0 \to +\infty$ due to the factorial damping, so that there is no longer ambiguity (the other possible choice of contour for the original Borel integral would obviously give the same result).
Moreover if $A \leq 1$, (21) is convergent, since the integrand behaves in such case *at most* as an exponential (being exactly exponential for $A = 1$), thus (Borel) integrable[8].
For the (majorant series) approximation (22), one can indeed integrate the Borel analytically, at least for particular values of A (e.g. integer $1/A$). For instance for the simplest RS choice $A = 1$, Eq. (21) becomes

$$\int_0^\infty dt e^{-t} \,[1 + \frac{e}{b_0}(e^{e\, t/m''} - 1)\,] = 1 + \frac{e}{b_0}\, \frac{e/m''}{1 - e/m''} \tag{23}$$

of convergence radius $1/m'' < e^{-1}$ [thus giving a *lower* bound for the actual convergence radius of the exact series of coefficients $d^B_{q,N}(A)$]. Similarly, for integer $1/A \geq 2$, (21) can be expressed in terms of known analytic functions and converges even for arbitrary $m'' > 0$.

6 Discussion

Though renormalon ambiguities are perturbative artifacts expected to disappear (or more precisely to cancel out with OPE contributions) in truly non-perturbative calculations[1, 2, 27], such explicit cancellations are generally beyond the scope for theories like QCD. Rather, the peculiar damping mechanism of factorial divergences exhibited here is intuitively due to the fact that our reorganization of perturbative expansions makes those much more similar to the oscillator energy levels expansion, exhibiting a dependence on $\hat{m}_v/\Lambda$, Eq. (4), which is *power-like* (rather than log-like) for sufficiently small $\hat{m}_v$, *and* the adjustable parameter $\hat{m}_v/\Lambda$ may be order-dependently rescaled. Note indeed that the linear DE-VIP taking the form (14), and (20) when combined with the Borel method, is only one among various similar resummation means. In particular, we emphasize that the obtained convergence properties do not depend on the detailed properties of the contour integrals here considered, e.g. Eq. (14) [which however have the advantage of giving rather simple and tractable expressions in the massless limit and for Borel transforms Eqs. (20)–(23)]: more generally performing a "brute force" δ-expansion of e.g. (7) and rescaling the trial mass m_v according to (13), replaces (15) and subsequent results with more complicated series [25], but with similar asymptotic and (Borel) convergence properties (and which, like in the oscillator case, can be equivalent to optimization in m_v at large delta-expansion orders). One may also further exploit the arbitrariness of m_v to construct[25] more general "deformations", or "order dependent mappings" [10] of the interaction terms which possibly lead more directly to improved convergence properties.

In summary, this construction may be considered an explicit counter-example to conventional wisdom arguments (i)–(iii) mentioned in introduction. In the present paper we have

[8] If $A > 1$, the integral (21) does not converge, so that the (majorant) series is not strictly Borel summable. But even in such case the absence of ambiguities (which usually call[2] for additional power corrections to the perturbative series) indicates that, from a physical point of view, there is no special transition at $A \geq 1$, i.e. no additional "non-perturbative" input to the (reorganized) series Eq (15) should be a priori needed.

analyzed only the formal Borel convergence properties, a priori applicable to any AFT, and which can be viewed as the generalization to $D > 1$ renormalizable theories of the ordinary convergence properties of the DE-VIP for the oscillator[9, 11]. Next we argue that such a summation recipe can provide a well-defined basis to estimate more precisely some of the χSB order parameters in QCD or other models, and a more detailed study with concrete numerical applications to the GN model and QCD will be explored in [25]. Though one may eventually raise that in QCD-like theories, other contributions to the χSB order parameters of "truly non-perturbative" origin (i.e. unreachable by any resummation mean, and/or related e.g. to instanton phenomena typically) may be expected, the resummation contributions here considered should be a useful piece of information.

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References

[1] F. David, Nucl. Phys. B209 (1982) 433; *ibid* B234 (1984) 237.

[2] For a recent extensive review see M. Beneke, Phys.Rept.317 (1999) 1.

[3] M. Shifman, A. Vainshtein, and V. Zakharov, Nucl. Phys. B147 (1979) 385.

[4] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345.

[5] D. J. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235.

[6] For a review on Borel summation method and many original references, see J. C. Le Guillou and J. Zinn-Justin eds., *Large-Order Behaviour of Perturbation Theory*, Current Physics–Sources and Comments, North-Holland, 1990.

[7] P. M. Stevenson, Phys. Rev. D23 (1981) 2916; Nucl. Phys. B203 (1982) 472.

[8] V. Yukalov, Teor. Mat. Fiz. 28(1976) 92; W.E. Caswell, Ann. Phys. (N.Y) 123 (1979) 153; I.G. Halliday and P. Suranyi, Phys. Lett. B85 (1979) 421; J. Killinbeck, J. Phys. A 14 (1981) 1005; R.P. Feynman and H. Kleinert, Phys. Rev. A34 (1986) 5080; A. Okopinska, Phys. Rev. D 35 (1987) 1835; A. Duncan and M. Moshe, Phys. Lett. 215 B (1988) 352; H.F. Jones and M. Moshe, Phys. Lett. 234 B (1990) 492; A. Neveu, Nucl. Phys. B18(1990) (Proc. Suppl.) 242; V. Yukalov, J. Math. Phys 32(1991) 1235; S. Gandhi, H.F. Jones and M. Pinto, Nucl. Phys. B 359 (1991) 429; C. M. Bender et al., Phys. Rev. D45 (1992) 1248; S. Gandhi and M. Pinto, Phys. Rev. D46(1992) 2570; H. Yamada, Z. Phys. C59 (1993) 67; A.N. Sissakian, I.L. Solovtsov and O.P. Solovtsova, Phys. Lett. B321 (1994) 381; H. Kleinert, Phys. Rev. D57 (1998) 2264; Phys. Lett. B 434, 74 (1998); see also *Critical Properties of* ϕ^4*-Theories*, H. Kleinert and V. Schulte-Frohlinde, World Scientific (2001), chap. 19 for a recent review.

[9] R. Guida, K. Konishi and H. Suzuki, Ann. Phys. 241 (1995) 152; Annals Phys. 249 (1996) 109.

[10] R. Seznec and J. Zinn-Justin, J. Math. Phys. 20 (1979) 1398; J.C. Le Guillou and J. Zinn-Justin, Ann. Phys. 147 (1983) 57.

[11] A. Duncan and H.F. Jones, Phys. Rev. D47 (1993) 2560; C.M. Bender, A. Duncan and H.F. Jones, Phys. Rev. D 49 (1994) 4219; B. Bellet, P. Garcia and A. Neveu, Int.J.Mod.Phys. A11(1996) 5587; and 5607; C. Arvanitis, H.F. Jones and C. Parker, Phys. Rev. D 52 (1995) 3704; H. Kleinert, Phys. Rev. Lett.75 (1995) 2787; W. Janke and H. Kleinert, Phys. Lett. A206 (1995) 283.

[12] C.M. Bender and T.T. Wu, Phys. Rev. 184 (1969) 1231; Phys. Rev. D7 (1973) 1620.

[13] C. Arvanitis et al, Int.J.Mod.Phys. A12 (1997) 3307.

[14] C. Arvanitis, F. Geniet, J.-L. Kneur and A. Neveu, Phys.Lett. B390 (1997) 385.

[15] J.-L. Kneur, Phys.Rev. D57 (1998) 2785.

[16] J.-L. Kneur and D. Reynaud, talks at QCD Euroconference, July 2000, Montpellier, and ICHEP2000, Aug. 2000, Osaka, to appear in the proceedings.

[17] P. Forgacs, F. Niedermayer and P. Weisz, Nucl. Phys. B 367 (1991) 123; 157.

[18] For a recent review on theoretical status of χSB see e.g. A. Smilga, hep-ph/0010049 and refs. therein.

[19] See e.g. R. Corless et al, Advances in Computational Mathematics, 5 (1996) 329.

[20] E. Gardi, G. Grunberg and M. Karliner, JHEP (1998) 9807:007

[21] D. Broadhurst, N. Gray, W. Grafe and K. Schilcher, Z. Phys. C48 (1990) 673.

[22] J.A. Gracey, Nucl. Phys. B 341 (1990) 403; *ibid.* B 367 (1991) 657; C. Luperini and P. Rossi, Ann. Phys. (N.Y.) 212 (1991) 371.

[23] R. Tarrach, Nucl. Phys. B183 (1981) 384; A. Kronfeld, Phys. Rev. D58 (1998) 051501.

[24] M. Beneke and V. Braun, Nucl. Phys. B426 (1994) 301; I. Bigi et al, Phys. Rev. D50 (1994) 2234.

[25] J.-L. Kneur and D. Reynaud, in preparation.

[26] D. Reynaud, thesis in preparation.

[27] M. Beneke, V.M. Braun and N. Kivel, Phys. Lett. B443 (1998) 308.

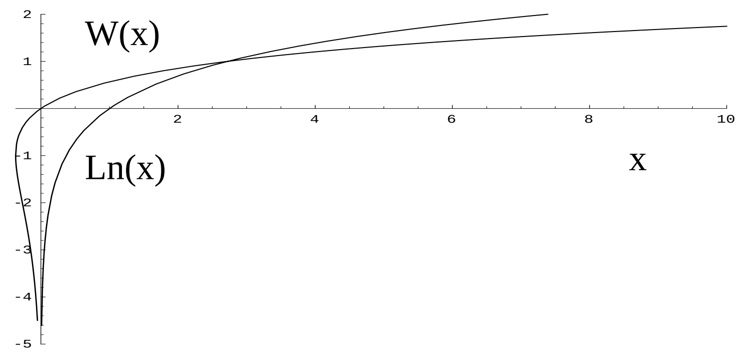

Figure 1: The Lambert W function compared to the Log.

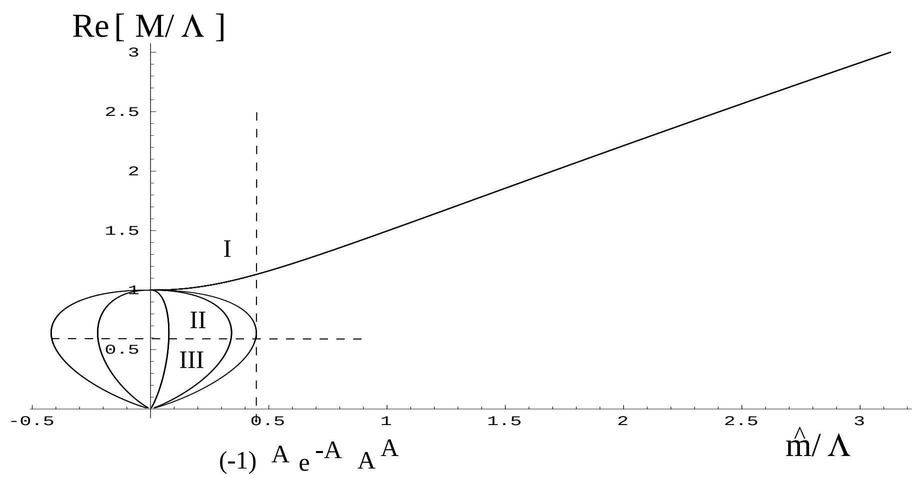

Figure 2: The different branches of (the real part of) M_1 in Eq. (2), for $A = 4/9$ (corresponding to first RG order QCD with three active quark flavours).

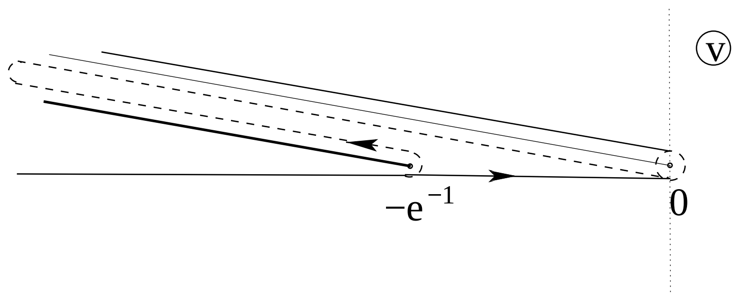

Figure 3: Singularities and equivalent integration contours in the v plane, for $A = 1$.

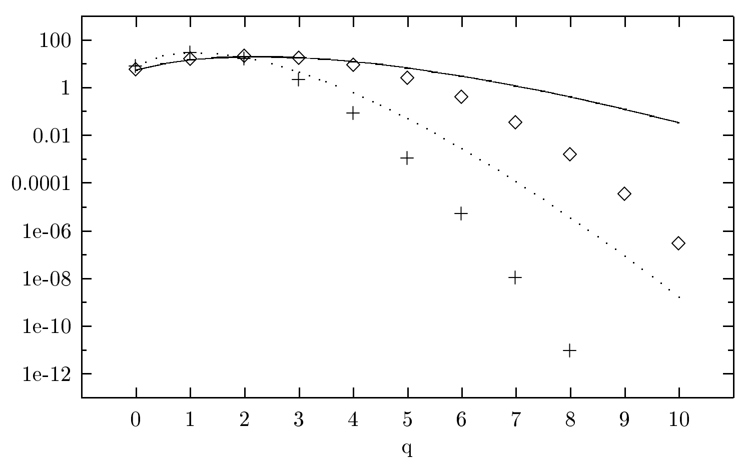

Figure 4: Comparison between exact (diamonds and crosses) and $N \to \infty$ (full and dotted lines) $d^B_{q,N}(A)$ coefficients of Borel integrand Eq. (21) for $A = 1$ and $A = 1/2$ respectively.